Lecture 13. Vector spaces

Def (1) A vector space is a set endowed with well-defined addition and scalar multiplication.

(2) An element of a vector space V is called a vector in V

Note (1) $\mathbb{R}^n$ is a vector space.

* We will eventually learn that every vector space is essentially equal to $\mathbb{R}^n$

(2) In most cases (and always in Math 313), the addition and scalar multiplication of a vector space are defined in an obvious manner.

(3) If the addition and scalar multiplication are evidently defined, a vector space is a set V with the following properties:

- $\vec{0} \in V$ (containing the zero vector)
- $\vec{u}, \vec{v} \in V \Rightarrow \vec{u} + \vec{v} \in V$ (closed under addition)
- $c \in \mathbb{R}, \vec{v} \in V \Rightarrow c\vec{v} \in V$ (closed under scalar multiplication)

(4) A vector space cannot be an empty set, as it contains at least one element, namely the zero vector.

Ex Determine whether each set is a vector space.

(1) The set of all points on the line $y=x$ in $\mathbb{R}^2$.

Sol • zero vector : $(0,0)$ on the line $y=x$

• closed under addition :

$\vec{u}=(a,a)$ and $\vec{v}=(b,b)$ on the line $y=x$

$\Rightarrow \vec{u}+\vec{v}=(a+b,\, a+b)$ on the line $y=x$

• closed under scalar multiplication :

$\vec{v}=(a,a)$ on the line $y=x$

$\Rightarrow c\vec{v}=(ca,ca)$ on the line $y=x$ for any $c\in\mathbb{R}$

Hence the set is $\boxed{\text{a vector space}}$

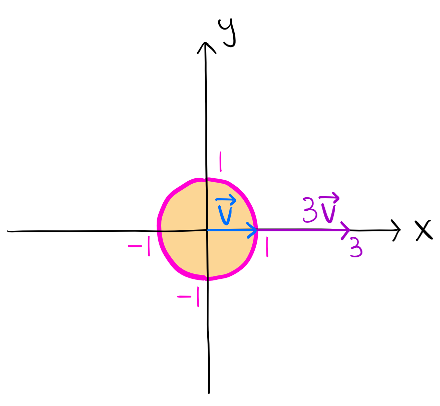
(2) The set of all points on the line $y=x+2$ in $\mathbb{R}^2$.

Sol The zero point $(0,0)$ is not on the line $y=x+2$

Hence the set is $\boxed{\text{not a vector space}}$

(3) The set of all points (x,y) in $\mathbb{R}^2$ with $x^2+y^2\le 1$.

Sol



The set is not closed under scalar multiplication :

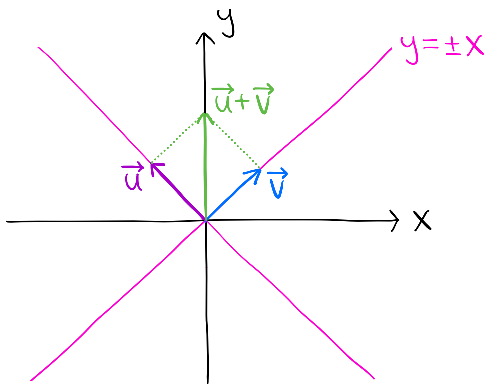
$\begin{cases} \vec{v}=(1,0) \text{ satisfying } x^2+y^2\le 1 \\ 3\vec{v}=(3,0) \text{ not satisfying } x^2+y^2\le 1 \end{cases}$

Hence the set is $\boxed{\text{not a vector space}}$

Note In general, every bounded set is not a vector space.

(4) The set of all points (x,y) in $\mathbb{R}^2$ with $x^2=y^2$.

Sol



We have $x^2=y^2 \iff y=\pm x$

The set is not closed under addition:

$\begin{cases} \vec{u}=(-1,1) \text{ and } \vec{v}=(1,1) \text{ on the set} \\ \vec{u}+\vec{v}=(0,2) \text{ not on the set} \end{cases}$

Hence the set is $\boxed{\text{not a vector space}}$

(5) The set of all polynomials of degree 2.

Sol The zero polynomial does not have degree 2.

Hence the set is $\boxed{\text{not a vector space}}$

Note Depending on the convention, the degree of the zero polynomial can be $0, -\infty$, or undefined. In Math 313, we take it to be 0.

(6) The set of all polynomials of degree at most 2.

Sol
- zero vector: the zero polynomial of degree $0 \le 2$
- closed under addition:

  $p(t)=a_0+a_1t+a_2t^2$ and $q(t)=b_0+b_1t+b_2t^2$ in the set

  $\implies p(t)+q(t)=(a_0+b_0)+(a_1+b_1)t+(a_2+b_2)t^2$ in the set
- closed under scalar multiplication:

  $p(t)=a_0+a_1t+a_2t^2$ in the set

  $\implies cp(t)=ca_0+ca_1t+ca_2t^2$ in the set for any $c\in\mathbb{R}$

Hence the set is $\boxed{\text{a vector space}}$

(7) The set of all functions $f: \mathbb{R} \longrightarrow \mathbb{R}$ which are continuous.

Sol
- zero vector : the zero function in the set

  * The zero function is the constant function which returns 0 for any input value.

- closed under addition :

  $f: \mathbb{R} \longrightarrow \mathbb{R}$ and $g: \mathbb{R} \longrightarrow \mathbb{R}$ continuous

  $\Longrightarrow f+g$ continuous

- closed under scalar multiplication :

  $f: \mathbb{R} \longrightarrow \mathbb{R}$ continuous

  $\Longrightarrow cf$ continuous for any $c \in \mathbb{R}$

Hence the set is $\boxed{\text{a vector space}}$

(8) The set of all functions $f: \mathbb{R} \longrightarrow \mathbb{R}$ which are continuous at $x=1$.

Sol
- zero vector : the zero function in the set

- closed under addition :

  $f: \mathbb{R} \longrightarrow \mathbb{R}$ and $g: \mathbb{R} \longrightarrow \mathbb{R}$ continuous at $x=1$

  $\Longrightarrow f+g$ continuous at $x=1$

- closed under scalar multiplication :

  $f: \mathbb{R} \longrightarrow \mathbb{R}$ continuous at $x=1$

  $\Longrightarrow cf$ continuous at $x=1$ for any $c \in \mathbb{R}$

Hence the set is $\boxed{\text{a vector space}}$